Chapter 1

Fuzzy Sets

This chapter begins with a brief review of classical sets in order to facilitate the introduction of fuzzy sets. Next the concept of membership function is explained. It defines the degree to which an element under consideration belongs to a fuzzy set. Fuzzy numbers are described as a particular case of fuzzy sets. Fuzzy sets and fuzzy numbers will be used in fuzzy logic to model words such as profit, investment, cost, income, age, etc. Fuzzy relations together with some operations on fuzzy relations are introduced as a generalization of fuzzy sets and ordinary relations. They have application in database models. Fuzzy sets and fuzzy relations play an important role in fuzzy logic.

1.1 Classical Sets: Relations and Functions

Classical sets

This section reviews briefly the terminology, notations, and basic properties of *classical sets*, usually called *sets*.

The concept of a *set* or *collection* of objects is common in our everyday experience. For instance, all persons listed in a certain telephone directory, all employees in a company, etc. There is a defining property that allows us to consider the objects as a whole. The objects in a set are called *elements* or *members* of the set. We will denote elements by small letters $a, b, c, \ldots, x, y, z$ and the sets by capital letters

$A, B, C, \ldots, X, Y, Z$. Sets are also called ordinary or crisp in order to be distinguished from fuzzy sets.

The fundamental notion in set theory is that of *belonging* or *membership*. If an object x belongs to the set A we write $x \in A$; if x is not a member of A, we write $x \notin A$. In other words for each object x there are only two possibilities: either x belongs to A or it does not.[1]

A set containing finite number of members is called *finite* set; otherwise it is called *infinite* set. We present two methods of describing sets.

Listing method

The set is described by *listing* its elements placed in braces; for example $A = \{1, 3, 6, 7, 8\}$, $B = \{\text{business, finance, management}\}$. The order in which elements are listed is of no importance. An element should be listed only once.

Membership rule

The set is described by one or more properties to be satisfied only by objects in the set:

$$A = \{x \mid x \text{ satisfies some property or properties}\}.$$

This reads: "A is the set of all x such that x satisfies some property or properties." For example $R = \{x \mid x \text{ is real number}\}$ reads: "R is the set of all x such that x is a real number"; $R_+ = \{x | x \geq 0, x \in R\}$ reads "R_+ is the set of all x which are nonnegative real numbers."

Universal set

The set of all objects under consideration in a particular situation is called *universal set or universe*; it will be denoted by U.

Empty set

A set without elements is called empty; it is denoted by ϕ.

Interval

The set of all real numbers x such that $a_1 \leq x \leq a_2$, where a_1 and a_2 are real numbers, form a closed interval $[a_1, a_2] = \{x \mid a_1 \leq x \leq a_2, x \in R\}$ with boundaries a_1 and a_2. It is also called *interval number*.

Equal sets

Sets A and B are *equal* , denoted by $A = B$, if they have the same elements.

Subset

The set A is a *subset* of the set B (A is *included* in B), denoted by $A \subseteq B$, if every element of A is also an element of B. Every set is subset of itself, $A \subseteq A$. The empty set ϕ is a subset of any set. It is assumed that each set we are dealing with is a subset of a universal set U.

Proper subset

A is a *proper subset* of B, denoted $A \subset B$, if $A \subseteq B$ and there is at least one element in B which does not belong to A. For instance $\{a, b\} \subset \{a, b, c\}$. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Intersection

The intersection of the sets A and B, denoted by $A \cap B$, is defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}; \tag{1.1}$$

$A \cap B$ is a set whose elements are common to A and B.

Union

The union of A and B , denoted by $A \cup B$, is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}; \tag{1.2}$$

$A \cup B$ is a set whose elements are in A or B, including any element that belongs to both A and B.

Disjoint sets

If the sets A and B have no elements in common, they are called *disjoint.*

Complement

The *complement* of $A \subset U$, denoted by $\overline{A}$, is the set

$$\overline{A} = \{x \in U \mid x \notin A\}. \tag{1.3}$$

The complement of a set consists of all elements in the universal set that are not in the given set.

Example 1.1

Given the sets

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}, \quad U = \{1, 2, 3, 4, 5, 6, 7\},$$

then using (1.1)–(1.3) we find

$$A \cap B = \{1, 3\}, \;\; A \cup B = \{1, 2, 3, 4, 5, 6\}, \;\; \overline{A} = \{5, 6, 7\}, \;\; \overline{B} = \{2, 4, 7\}.$$

□

Convex sets

Consider the universe U to be the set of real numbers R.

A subset S of R is said to be *convex* if and only if, for all $x_1, x_2 \in S$ and for every real number λ satisfying $0 \le \lambda \le 1$, we have

$$\lambda x_1 + (1 - \lambda)x_2 \in S.$$

For example, any interval $S = [a_1, a_2]$ is a convex set since the above condition is satisfied; [0, 1] and [3, 4] are convex, but $[0, 1] \cup [3, 4]$ is not.

Venn diagrams

Sets are geometrically represented by circles inside a rectangle (the universal set U). In Fig. 1.1 are shown the sets $A \cap B$ and $A \cup B$.

Fig. 1.1. Venn diagrams for $A \cap B$(intersection), $A \cup B$(union).

Ordered pairs

It was noted that the order of the elements of a set is not important. However there are cases when the order is important. To indicate that a set or pair of two elements a and b is *ordered*, we write (a, b), i.e. use parentheses instead of braces; a is called *first element* of the pair and b is called *second element*.

Cartesian product

Cartesian product (or *cross product*) of the sets A and B denoted $A \times B$ is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.4}$$

Example 1.2

(a) Given

$$A = \{1, 2, 3\}, \qquad B = \{1, 2\},$$

then according to (1.4) we find

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\};$$

geometrically it is presented on Fig. 1.2 (a).
(b) If $X, Y = R$, the set of all real numbers, then

$$X \times Y = \{(x, y)|x \in X, y \in Y\} = R \times R$$

is the set of all ordered pairs which form the cartesian plane xy (see Fig. 1.2(b)).

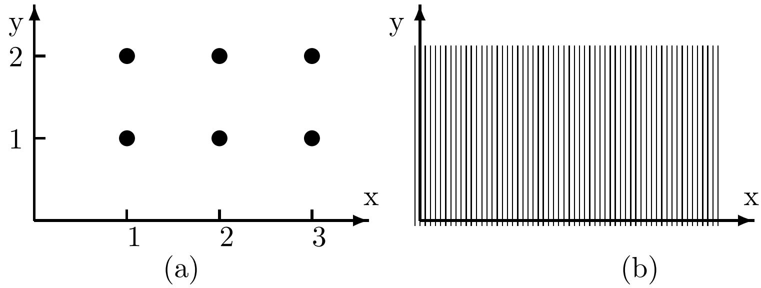


Fig. 1.2. (a) Cartesian product $\{1, 2, 3\} \times \{1, 2\}$; (b) Cartesian plane.

□

Relations

The concept of *relation* is very general. It is based on the concepts of ordered pair (a, b), $a \in A$, $b \in B$, and cartesian product of the sets A and B.

A *relation* from A to B (or between A and B) is any subset $\Re$ of the cartesian product $A \times B$. We say that $a \in A$ and $b \in B$ are related by $\Re$; the elements a and b form the *domain* and *range* of the relation, correspondingly. Since a relation is a set, it may be described by either the listing method or the membership rule. The relation $\Re$ is called *binary relation* since two sets, A and B, are related.

Example 1.3

Let $A = \{x_1, x_2, x_3\}$ and $B = \{1, 2, 3, 4\}$.
We list some binary relations generated by A and B:

$$\begin{aligned}
&\Re_1 = \{(x_1, 1), (x_2, 1), (x_3, 4)\},\\
&\Re_2 = \{(x_1, 2), (x_1, 3)\}, \quad \Re_3 = \{(x_2, 2), (x_3, 1)\},\\
&\Re_4 = \{(x_1, 1), (x_1, 2), (x_1, 3), (x_1, 4), (x_2, 1), (x_4, 1)\}
\end{aligned}$$

are relations from A to B;

$$\begin{aligned}
&\Re_5 = \{(1, x_2), (2, x_3), (3, x_1)\}, \quad \Re_6 = \{(1, x_1), (2, x_1)\},\\
&\Re_7 = \{(1, x_1), (1, x_2), (1, x_4)\}, \quad \Re_8 = \{(2, x_1), (3, x_3)\}
\end{aligned}$$

are relations from B to A; the empty set ϕ is a relation; the cross product $A \times B$ is a relation from A to B and the cross product $B \times A$ is a relation from B to A.

□

Functions

A *function* f is a relation $\Re$ such that for every element x in the domain of f there corresponds a unique element y in the range of f. For instance the relations in Example 1.2 are not functions.

We often say that f *maps* x onto y; y is the *image* of x under f. Then we can write $f : x \to y$. However, it is customary to use the notation $y = f(x)$.

Generalization

The notions of ordered pair, Cartesian product, relation, and function can be generalized for higher dimensions than two. For instance when $n = 3$ we have:

Ordered triple (a, b, c);

Cartesian product

$$A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\};$$

Relation from $A \times B$ to C is any subset $\Re$ of $A \times B \times C$.

Function $z = f(x, y)$ is a relation such that for every pair (x, y) in the domain of f there corresponds a unique element z in its range.

Characteristic Function

The membership rule that characterizes the elements (members) of a set $A \subset U$ can be established by the concept of *characteristic function* (or *membership function*) $\mu_A(x)$ taking only two values, 1 and 0, indicating whether or not $x \in U$ is a member of A:

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad x \in A, \\ 0 & \text{for} \quad x \notin A. \end{cases} \tag{1.5}$$

Hence $\mu_A(x) \in \{0, 1\}$. Inversely, if a function $\mu_A(x)$ is defined by (1.5), then it is the characteristic function for a set $A \subset U$ in the sense that A consists of the values of $x \in U$ for which $\mu_A(x)$ is equal to 1. In other words every set is uniquely determined by its characteristic function.

The universal set U has for membership function $\mu_U(x)$ which is identically equal to 1, i.e. $\mu_U(x) = 1$. The empty set ϕ has for membership function $\mu_\phi(x) = 0$.

Example 1.4

Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and its subset A,

$$A = \{x_2, x_3, x_5\}.$$

Only three of the six elements in U belong A. Using the notation (1.5) gives

$$\mu_A(x_1) = 0, \;\; \mu_A(x_2) = 1, \;\; \mu_A(x_3) = 1,$$
$$\mu_A(x_4) = 0, \;\; \mu_A(x_5) = 1, \;\; \mu_A(x_6) = 0.$$

Hence the characteristic function of the set A is

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad x = x_2, x_3, x_5, \\ 0 & \text{for} \quad x = x_1, x_4, x_6; \end{cases}$$

The set A can be represented as

$$A = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1), (x_6, 0)\}.$$

□

Example 1.5

Let us try to use crisp sets to describe *tall men*. Consider for instance a man as tall if his height is 180 cm or greater; otherwise the man is not tall. The characteristic function of the set $A = \{\text{tall men}\}$ then is

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad 180 \le x, \\ 0 & \text{for} \quad 160 \le x < 180. \end{cases}$$

It is shown in Fig. 1.3, where the universe is $U = \{x \mid 160 \le x \le 200\}$.

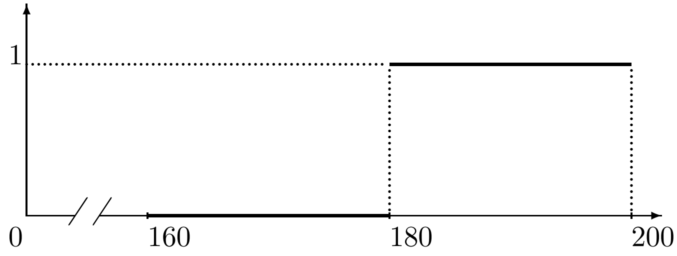


Fig. 1.3. Membership function of the set *tall men*.

Clearly this description of the set of *tall men* is not satisfactory since it does not allow gradation. The word *tall* is vague. For instance, a person whose height is 179 cm is not tall as well as a person whose height is 160 cm. Yet a person whose height is 180 is tall and so is a person with height 200 cm. Also the above definition introduces a drastic difference between heights of 179 cm and 180 cm, thus fails to describe realistically borderline cases.[2]

□

The concept of characteristic function introduced here will facilitate the understanding of the concept *fuzzy set*, the subject of the next section.

1.2 Definition of Fuzzy Sets

We have seen that belonging or membership of an object to a set is a precise concept; the object is either a member to a set or it is not, hence the membership function can take only two values, 1 or 0. The set *tall men* in Example 1.5 illustrates the need to increase the describing capabilities of classical sets while dealing with words.

To describe gradual transitions Zadeh (1965), the founder of fuzzy sets, introduced grades between 0 and 1 and the concept of graded membership.

Let us refer to Example 1.4. Each of the six elements of the universal set $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ either belongs to or does not belong to the set $A = \{x_2, x_3, x_5\}$. According to this, the characteristic function $\mu_A(x)$ takes only the values 1 or 0. Assume now that a characteristic function may take values in the interval [0, 1]. In this way the concept of membership is not any more *crisp* (either 1 or 0), but becomes *fuzzy* in the sense of representing partial belonging or *degree of membership.*

Consider a classical set A of the universe U. A *fuzzy set* $\mathcal{A}$ is defined by a set or ordered pairs, a binary relation,

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \mid x \in A, \mu_{\mathcal{A}}(x) \in [0, 1]\}, \tag{1.6}$$

where $\mu_{\mathcal{A}}(x)$ is a function called *membership function*; $\mu_{\mathcal{A}}(x)$ specifies the *grade* or *degree* to which any element x in A belongs to the fuzzy set $\mathcal{A}$. Definition (1.6) associates with each element x in A a real number $\mu_{\mathcal{A}}(x)$ in the interval [0, 1] which is assigned to x. Larger values of $\mu_{\mathcal{A}}(x)$ indicate higher degrees of membership.[3]

Let us express the meaning of (1.6) in a slightly modified way. The first elements x in the pair $(x, \mu_{\mathcal{A}}(x))$ are given numbers or objects of the classical set A; they satisfy some property (P) under consideration partly (to various degrees). The second elements $\mu_{\mathcal{A}}(x)$ belong to the interval (classical set) $[0, 1]$; they indicate to what extent (degree) the elements x satisfy the property P.

It is assumed here that the membership function $\mu_{\mathcal{A}}(x)$ is either piecewise continuous or discrete.

The fuzzy set $\mathcal{A}$ according to definition (1.6) is formally *equal* to its membership function $\mu_{\mathcal{A}}(x)$. We will *identify* any fuzzy set with its membership function and use these two concepts as *interchangeable.* Also we may look at a fuzzy set over a domain A as a function mapping A into [0, 1].

Fuzzy sets are denoted by italic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and the corresponding membership functions by $\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x), \mu_{\mathcal{C}}(x), \ldots$.

Elements with zero degree of membership in a fuzzy set are usually not listed.

Classical sets can be considered as a special case of fuzzy sets with all membership grades equal to 1.

A fuzzy set is called *normalized* when at least one $x \in A$ attains the maximum membership grade 1; otherwise the set is called *nonnormalized.* Assume the set $\mathcal{A}$ is nonnormalized; then $\max \mu_{\mathcal{A}}(x) < 1$. To normalize the set $\mathcal{A}$ means to normalize its membership function $\mu_{\mathcal{A}}(x)$, i.e. to divide it by $\max \mu_{\mathcal{A}}(x)$, which gives $\frac{\mu_{\mathcal{A}}(x)}{\max \mu_{\mathcal{A}}(x)}$.

$\mathcal{A}$ is called empty set labeled ϕ if $\mu_{\mathcal{A}}(x) = 0$ for each $x \in A$.

The fuzzy set $\mathcal{A} = \{(x_1, \mu_{\mathcal{A}}(x_1))\}$, where x_1 is the only value in $A \subset U$ and $\mu_{\mathcal{A}}(x_1) \in [0, 1]$, is called *fuzzy singleton.*

While the set A is a subset of the universal set U which is crisp, the fuzzy set $\mathcal{A}$ is not.

Instead of (1.6), some authors use the notation

$$\mathcal{A} = \{\mu_{\mathcal{A}}(x)/x, x \in A, \mu_{\mathcal{A}}(x) \in [0, 1]\},$$

where the symbol / is not a division sign but indicates that the top number $\mu_{\mathcal{A}}(x)$ is the membership value of the element x in the bottom.

Example 1.6

Consider the fuzzy set

$$\mathcal{A} = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.3), (x_4, 0.8), (x_5, 1), (x_6, 0.2)\}$$

which also can be represented as

$$\mathcal{A} = 0.1/x_1 + 0.5/x_2 + 0.3/x_3 + 0.8/x_4 + 1/x_5 + 0.2/x_6;$$

it is a discrete fuzzy set consisting of six ordered pairs. The elements $x_i, i = 1, \ldots, 6$, are not necessary numbers; they belong to the classical set $A = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ which is a subset of a certain universal set U. The membership function $\mu_{\mathcal{A}}(x)$ of $\mathcal{A}$ takes the following values on [0, 1]:

$$\begin{array}{lll}\mu_{\mathcal{A}}(x_1) = 0.1, & \mu_{\mathcal{A}}(x_2) = 0.5, & \mu_{\mathcal{A}}(x_3) = 0.3, \\ \mu_{\mathcal{A}}(x_4) = 0.8, & \mu_{\mathcal{A}}(x_5) = 1, & \mu_{\mathcal{A}}(x_6) = 0.2.\end{array}$$

The following interpretation could be given to $\mu_{\mathcal{A}}(x_i), i = 1, \cdots, 6$. The element x_5 is a *full* member of the fuzzy set $\mathcal{A}$, while the element x_1 is a member of $\mathcal{A}$ a *little* ($\mu_A(x_1) = 0.1$ is near 0); x_6 and x_3 are a *little more* members of $\mathcal{A}$; the element x_4 is *almost* a full member of $\mathcal{A}$, while x_2 is *more or less* a member of $\mathcal{A}$.

The fuzzy set $\mathcal{A}$ can be given also by the table

$$\mathcal{A} \triangleq \begin{array}{|llllll} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 0.1 & 0.5 & 0.3 & 0.8 & 1 & 0.2 \end{array}$$

where the symbol $\triangleq$ means "is defined by."

Now we specify in two different ways the elements x_i in A:

(a) Assume that $x_i, i = 1, \cdots, 6$, are integers, namely, $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$; they belong to the set $A = \{1, 2, 3, 4, 5, 6\}$, a subset of the universe $U = N$, the set of all integers. The fuzzy set $\mathcal{A}$ becomes

$$\mathcal{A} = \{(1, 0.1), (2, 0.5), (3, 0.3), (4, 0.8), (5, 1), (6, 0.2)\};$$

its membership function $\mu_{\mathcal{A}}(x)$ shown in Fig. 1.4 by dots is a discrete one.

(b) Assume now that $x_i, i = 1, \ldots, 6$, are friends of George whose names are as follows: x_1 is Ron, x_2 is Ted, x_3 is John, x_4 is Joe, x_5 is Tom, and x_6 is Sam. They form a set of friends of George,

$$A = \{\text{Ron, Ted, John, Joe, Tom, Sam}\},$$

a subset of the universe U (all friends of George). The fuzzy set $\mathcal{A}$ here expresses *closeness of friends* of George on $A \subseteq U$:

$$\mathcal{A} = \{(\text{Ron}, 0.1), (\text{Ted}, 0.5), (\text{John}, 0.3), (\text{Joe}, 0.8), (\text{Tom}, 1), (\text{Sam}, 0.2)\}.$$

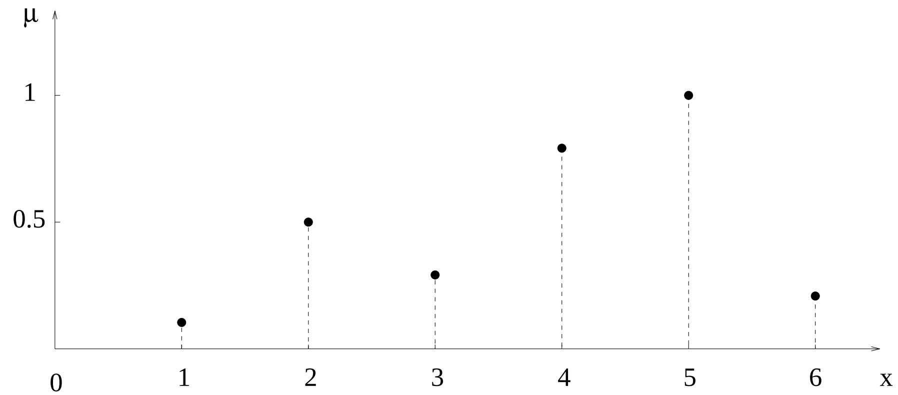


Fig. 1.4. Fuzzy set $\mathcal{A} = \{(1, 0.1), (2, 0.5), (3, 0.3), (4, 0.8), (5, 1), (6, 0.2)\}$.

$\square$

Example 1.7

Let us describe numbers *close* to 10.
(a) First consider the fuzzy set

$$\mathcal{A}_1 = \{(x, \mu_{\mathcal{A}_1}(x)) \mid x \in [5, 15], \mu_{\mathcal{A}_1}(x) = \frac{1}{1 + (x - 10)^2}\},$$

where $\mu_{\mathcal{A}_1}(x)$ shown in Fig. 1.5 is a continuous function.

The fuzzy set $\mathcal{A}_1$ represents real numbers *close* to 10.

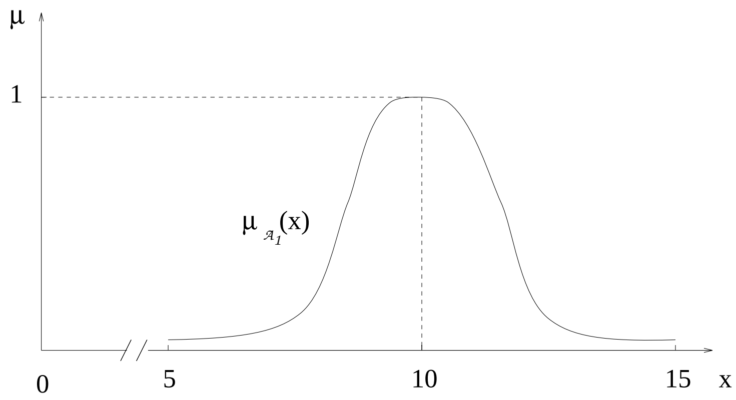


Fig. 1.5. Real numbers *close* to 10.

(b) Integers *close* to 10 can be expressed by the finite fuzzy set consisting of seven ordered pairs

$$\mathcal{A}_2 = \{(7, 0.1), (8, 0.3), (9, 0.8), (10, 1), (11, 0.8), (12, 0.3), (13, 0.1)\}.$$

The membership function of $\mathcal{A}_2$ is shown on Fig 1.6 by dots; it is a discrete function.

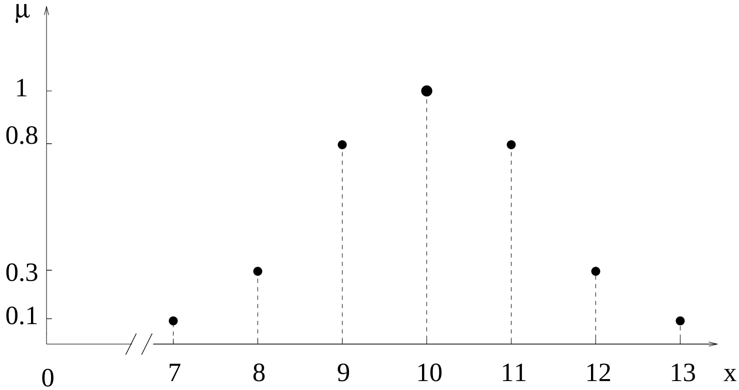


Fig. 1.6. Integers *close* to 10.

□

Example 1.8

We have seen in Example 1.5 that the description of *tall men* by classical sets is not adequate. Now we employ for the same purpose the fuzzy set $\mathcal{T} = \{(x, \mu_{\mathcal{T}}(x))\}$, where x measured in cm belongs to the interval [160, 200] and $\mu_{\mathcal{T}}(x)$ is defined by (see Fig 1.7)

$$\mu_{\mathcal{T}}(x) = \begin{cases} \frac{1}{2(30)^2}(x-140)^2 & \text{for} \quad 160 \leq x \leq 170, \\ -\frac{1}{2(30)^2}(x-200)^2+1 & \text{for} \quad 170 \leq x \leq 200. \end{cases}$$

The membership function $\mu_{\mathcal{T}}(x)$ is a continuous piecewise-quadratic function. The numbers on the horizontal axis x give height in cm and the vertical axis μ shows the degree to which a man can be labeled *tall.* According to the graph in Fig. 1.7, if a person's height is 160 cm, the person is a little tall (degree 0.22), 180 cm stands for almost tall (degree

0.78), 200 cm for tall (degree 1). The segment $[0.22, 1]$ of the vertical axis μ expresses the *quantification* of the degree of vagueness of the word *tall.*[4]

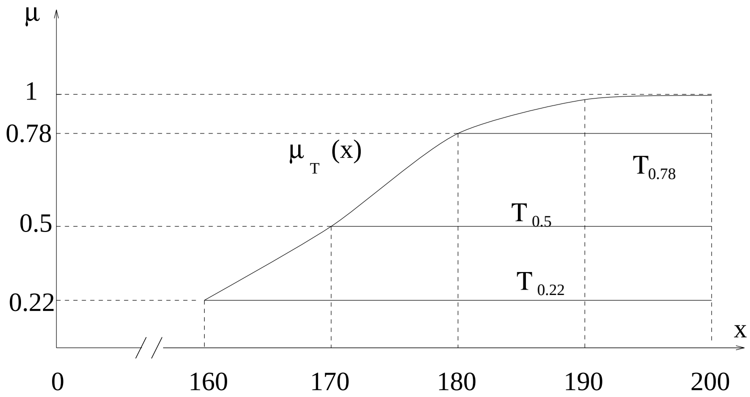


Fig. 1.7. Description of *tall men* by fuzzy set.

□

Further we define *α-level interval* or *α-cut*, denoted by A_α, as the crisp set of elements x which belong to $\mathcal{A}$ at least to the degree α:

$$A_\alpha = \{x \mid x \in R, \mu_{\mathcal{A}}(x) \geq \alpha\}, \quad \alpha \in [0, 1]. \tag{1.7}$$

It gives a *threshold* which provides a *level of confidence* α in a decision or concept modeled by a fuzzy set. We may use the threshold to discard from consideration those element x in A with grades of membership $\mu_{\mathcal{A}}(x) < \alpha$.

Example 1.9

Consider Example 1.8, the set $\mathcal{T}$, *tall men.* It has an infinite number of *α-level* intervals (*α-cuts*) denoted by $\mathcal{T}_\alpha$ where α varies between 0.22 and 1. Some α-cuts shown in Fig. 1.7 are given below:

$$\mathcal{T}_{0.22} = \{x|x \in R, 160 \leq x \leq 200\}, \mu_{\mathcal{T}}(x) \geq 0.22,$$
$$\mathcal{T}_{0.5} = \{x|x \in R, 170 \leq x \leq 200\}, \mu_{\mathcal{T}}(x) \geq 0.5,$$
$$\mathcal{T}_{0.78} = \{x|x \in R, 180 \leq x \leq 200\}, \mu_{\mathcal{T}}(x) \geq 0.78$$

For instance we may choose as a threshold the α-*cut* $\mathcal{T}_{0.5}$ thus discarding from consideration men whose height is below 170 cm.

□

A fuzzy set $\mathcal{A}$, where the universe $U = R$, is *convex* if and only if the α-level intervals $\mathcal{A}_\alpha$ (see (1.7)) are convex for all α in the interval $(0, 1]$. In such a case all α-*level* intervals $\mathcal{A}_\alpha$ consist of one segment (see Fig. 1.8(a)). Otherwise the set is nonconvex (see Fig. 1.8(b)).

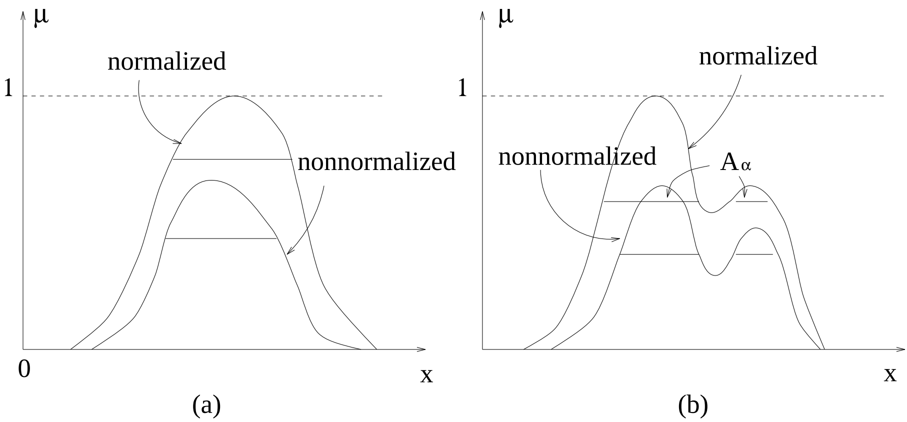


Fig. 1.8. (a) Convex fuzzy sets; (b) Nonconvex fuzzy sets.

1.3 Basic Operations on Fuzzy Sets

Consider the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ in the universe U,

$$\begin{aligned} \mathcal{A} &= \{(x, \mu_\mathcal{A}(x))\}, \quad \mu_\mathcal{A}(x) \in [0,1], \\ \mathcal{B} &= \{(x, \mu_\mathcal{B}(x))\}, \quad \mu_\mathcal{B}(x) \in [0,1]. \end{aligned}$$

The operations with $\mathcal{A}$ and $\mathcal{B}$ are introduced via operations on their membership functions $\mu_\mathcal{A}(x)$ and $\mu_\mathcal{B}(x)$.

Equality

The fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ are *equal* denoted by $\mathcal{A} = \mathcal{B}$ if and only if for every $x \in U$,

$$\mu_\mathcal{A}(x) = \mu_\mathcal{B}(x).$$

Inclusion

The fuzzy set $\mathcal{A}$ is *included* in the fuzzy set $\mathcal{B}$ denoted by $\mathcal{A} \subseteq \mathcal{B}$ if for every $x \in U$,

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x).$$

Then $\mathcal{A}$ is called a *subset* of $\mathcal{B}$.

Proper subset

The fuzzy set $\mathcal{A}$ is called a proper subset of the fuzzy set $\mathcal{B}$ denoted $\mathcal{A} \subset \mathcal{B}$ when $\mathcal{A}$ is a subset of $\mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, that is

$$\left.\begin{array}{ll} \mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x) & \text{for every } x \in U, \\ \mu_{\mathcal{A}}(x) < \mu_{\mathcal{B}}(x) & \text{for at least one } x \in U. \end{array}\right\}$$

For instance the nonnormalized sets in Fig. 1.8 (a) and (b) are proper.

Complementation

The fuzzy sets $\mathcal{A}$ and $\overline{\mathcal{A}}$ are complementary if

$$\mu_{\overline{\mathcal{A}}}(x) = 1 - \mu_{\mathcal{A}}(x) \quad \text{or} \quad \mu_{\mathcal{A}}(x) + \mu_{\overline{\mathcal{A}}}(x) = 1. \tag{1.8}$$

The membership function $\mu_{\overline{\mathcal{A}}}(x)$ is symmetrical to $\mu_{\mathcal{A}}(x)$ with respect to the line $\mu = 0.5$.

Intersection

The operation *intersection* of $\mathcal{A}$ and $\mathcal{B}$ denoted as $\mathcal{A} \cap \mathcal{B}$ is defined by

$$\mu_{\mathcal{A}\cap\mathcal{B}}(x) = \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)), \quad x \in U. \tag{1.9}$$

If $a_1 < a_2$, $\min(a_1, a_2) = a_1$. For instance $\min(0.5, 0.7) = 0.5$.

Union

The operation *union* of $\mathcal{A}$ and $\mathcal{B}$ denoted as $\mathcal{A} \cup \mathcal{B}$ is defined by

$$\mu_{\mathcal{A}\cup\mathcal{B}}(x) = \max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)), \quad x \in U. \tag{1.10}$$

If $a_1 < a_2$, $\max(a_1, a_2) = a_2$. For instance $\max(0.5, 0.7) = 0.7$.

Example 1.10

Consider the universe $U = \{x_1, x_2, x_3, x_4\}$ and the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ defined by the table

x	x_1	x_2	x_3	x_4
$\mu_A(x)$	0.2	0.7	1	0
$\mu_B(x)$	0.5	0.3	1	0.1

Using (1.9) and (1.10) gives

x	x_1	x_2	x_3	x_4
$\mu_{A\cap B}(x)$	0.2	0.3	1	0
$\mu_{A\cup B}(x)$	0.5	0.7	1	0.1

□

Schematic representation of operations on fuzzy sets

Fuzzy sets are schematically represented by their membership functions (assumed continuous) inside of rectangles. In Fig. 1.9 are shown $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(x)$, in Fig. 1.10 the complements $\mu_{\overline{\mathcal{A}}}(x)$ and $\mu_{\overline{\mathcal{B}}}(x)$, and in Fig. 1.11 the union $\mu_{\mathcal{A}\cap\mathcal{B}}(x)$ and the intersection $\mu_{\mathcal{A}\cap\mathcal{B}}(x)$.

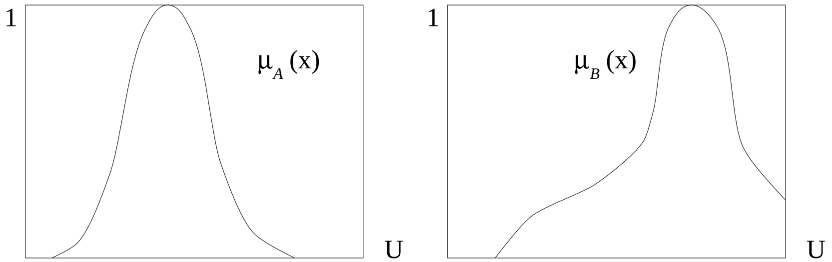


Fig. 1.9. Membership function $\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)$.

Figure 1.11 shows that $\mathcal{A} \cap \mathcal{B} \in \mathcal{A} \cup \mathcal{B}$.

Law of excluded middle and fuzzy sets

The classical sets possess an important property, the *law of excluded middle*,[2] expressed by $\mathcal{A} \cap \overline{\mathcal{A}} = \phi$ and $\mathcal{A} \cup \overline{\mathcal{A}} = U$. It is illustrated in Fig. 1.12 by the means of Venn diagrams.

The law of excluded middle is not valid for the fuzzy sets since $\mathcal{A} \cap \overline{\mathcal{A}} \neq \phi$ and $\mathcal{A} \cup \overline{\mathcal{A}} \neq U$. This is illustrated in Fig. 1.13.

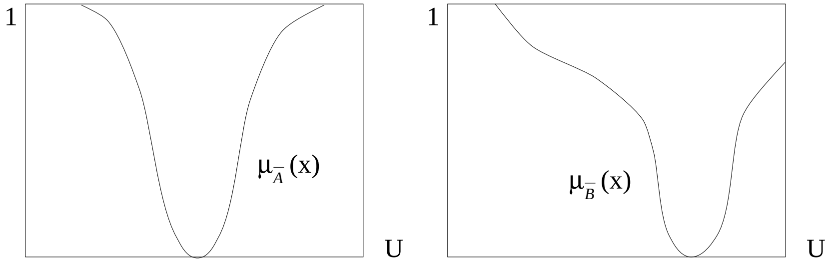


Fig. 1.10. Membership function $\mu_{\overline{\mathcal{A}}}(x), \mu_{\overline{\mathcal{B}}}(x)$.

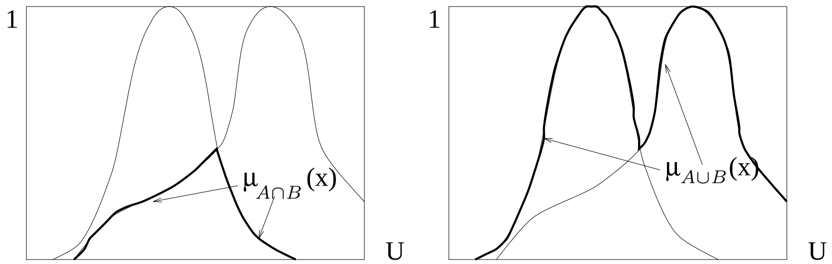
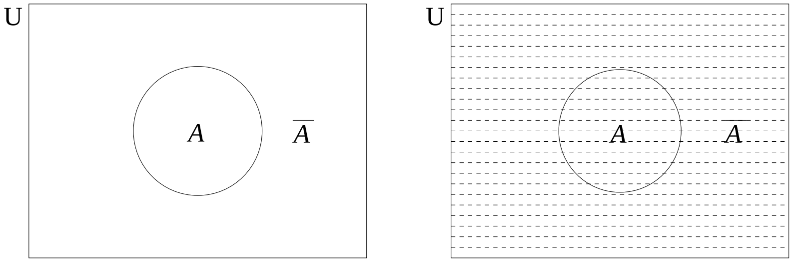


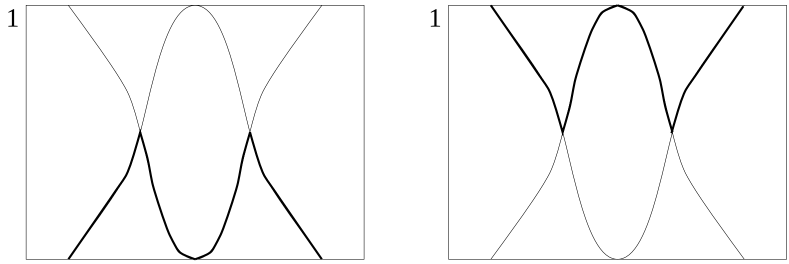
Fig. 1.11. Membership function of intersection and union.



$A \cap \overline{A} = \phi$ $\qquad$ $A \cup \overline{A} = U$

Fig. 1.12. The law of excluded middle for classical sets.

It is natural that the law of the excluded middle is not valid for fuzzy sets. In classical sets every object does or does not have a certain property, expressed by 1 or 0. Fuzzy sets were introduced to reflect the existence of objects in reality that have a property to a degree between 0 and 1. There are many shades of *gray* color between *black* and *white*.



$\mathcal{A} \cap \overline{\mathcal{A}} \neq \phi$ $\qquad$ $\mathcal{A} \cup \overline{\mathcal{A}} \neq U$

Fig. 1.13. The law of excluded middle is not valid for fuzzy sets.

The lack of the law of excluded middle in fuzzy set theory makes it less specific than that of classical set theory. However, at the same time, this lack makes fuzzy sets more general and flexible than classical sets and very suitable for describing vagueness and processes with incomplete and imprecise[3] information.

1.4 Fuzzy Numbers

A *fuzzy number*[5] is defined on the universe R as a *convex* and *normalized* fuzzy set. In Figs. 1.14(a),(b) are shown two fuzzy numbers, with a maximum and with a flat.

For instance, the normalized fuzzy set in Fig. 1.8(a) is a fuzzy number while the sets in Fig. 1.8(b) are not. The fuzzy set in Fig. 1.7 is also a fuzzy number.

The fuzzy set in Fig. 1.6 is a fuzzy number in the set of integers while the fuzzy set in Fig. 1.4 is not. Also we may consider a fuzzy number with a flat in the set of integers.

The interval $[a_1, a_2]$ is called *supporting interval* for the fuzzy number. For $x = a_M$ the fuzzy number in Fig. 1.14 (a) has a maximum. The flat segment (Fig. 1.14(b)) has maximum height 1; actually it is the *α-cut* at the highest confidence level 1.

Fuzzy numbers will be denoted by bold capital letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$, and their membership functions by $\mu_{\mathbf{A}}(x), \mu_{\mathbf{B}}(x), \mu_{\mathbf{C}}(x), \ldots$.

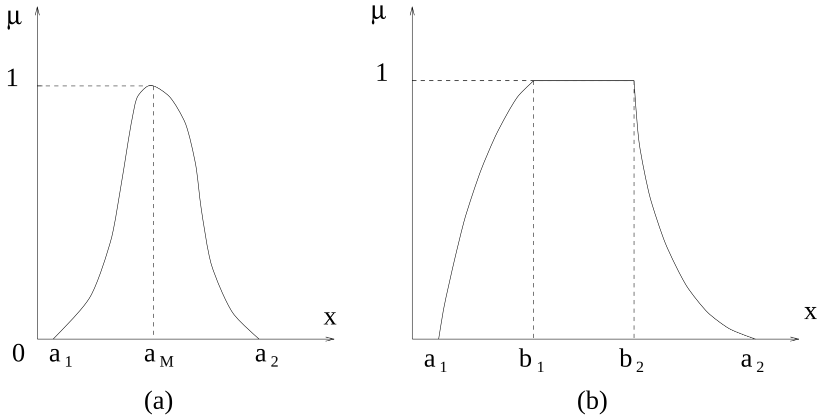


Fig. 1.14. Fuzzy numbers: (a) with a maximum; (b) with a flat.

Piecewise-quadratic fuzzy number

The membership function $\mu_{\mathbf{A}}(x)$ of a piecewise-quadratic fuzzy number shown in Fig. 1.15 is bell-shaped, symmetric about the line $x = p$, has a supporting interval $A = [a_1, a_2]$, and is characterized by two parameters, $p = \frac{1}{2}(a_1 + a_2)$ and $\beta \in (0, a_2 - p)$. The *peak-point* (the maximum point) is $(p, 1)$; 2β called *bandwidth* is defined as the segment (α-cut) at level $\alpha = \frac{1}{2}$ between the points $(p - \beta, \frac{1}{2})$ and $(p + \beta, \frac{1}{2})$, called *crossover points.*

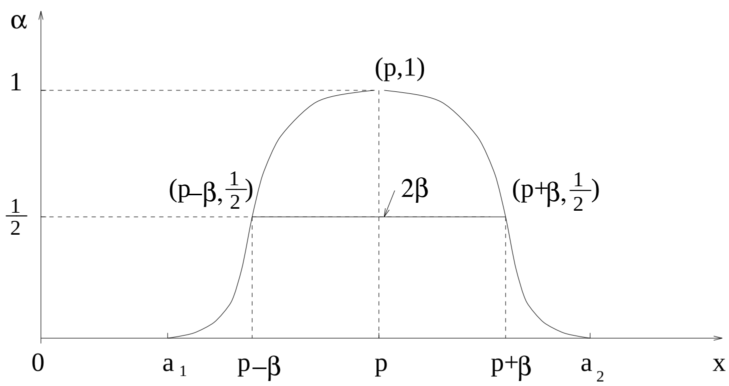


Fig. 1.15. Piecewise-quadratic fuzzy number.

The curve on Fig. 1.15 is described by the equations

$$\mu_{\mathbf{A}}(x) = \begin{cases} \frac{1}{2(p-\beta-a_1)^2}(x-a_1)^2 & \text{for } a_1 \le x \le p-\beta, \\ -\frac{1}{2\beta^2}(x-p)^2+1 & \text{for } p-\beta \le x \le p+\beta, \\ \frac{1}{2(p+\beta-a_2)^2}(x-a_2)^2 & \text{for } p+\beta \le x \le a_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}$$

The interpretation for the fuzzy number (1.11) is real numbers *close* to the number p. Since the word *close* is *vague* and in that sense *fuzzy*, it cannot be defined uniquely. That depends on the selection of the supporting interval and the bandwidth which are supposed to reflect a particular situation. For instance the fuzzy set *tall men* (Example 1.8) is a particular case of (1.11) (left branch) on the interval [160, 200] with $a_1 = 140, p = 200$, and $\beta = 30$.

Example 1.11

The manufacturing price of a product is *close* to 28. It can be described by the fuzzy number **A** in Fig. 1.16 where $a_1 = 23, a_2 = 33, p = 28, \beta = 3$.

The membership function $\mu_A(x)$ can be obtained from (1.11) by substituting the specific values of a_1, a_2, p and β given above.

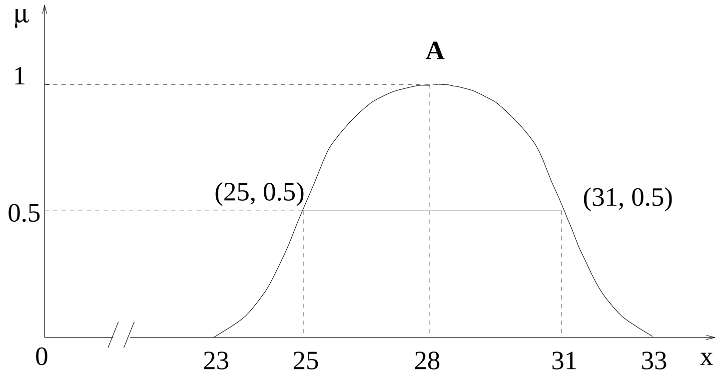


Fig. 1.16. Product price *close* to 28.

□

1.5 Triangular Fuzzy Numbers

A *triangular fuzzy number* **A** or simply triangular number with membership function $\mu_A(x)$ is defined on R by

$$\mathbf{A} \triangleq \mu_{\mathbf{A}}(x) = \begin{cases} \frac{x-a_1}{a_M-a_1} & \text{for } a_1 \le x \le a_M, \\ \frac{x-a_2}{a_M-a_2} & \text{for } a_M \le x \le a_2, \\ 0 & \text{otherwise,} \end{cases} \tag{1.12}$$

where $[a_1, a_2]$ is the supporting interval and the point $(a_M, 1)$ is the peak (see Fig. 1.17). The third line in (1.12) can be dropped.

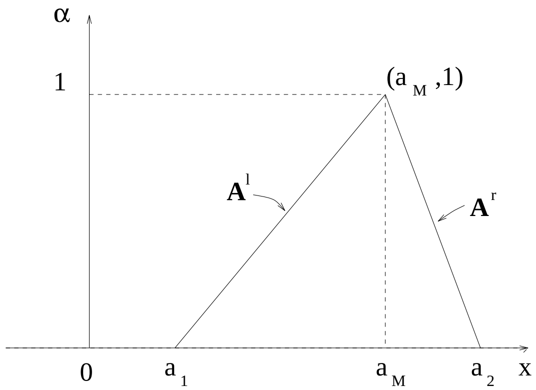


Fig. 1.17. Triangular fuzzy number.

Often in applications the point $a_M \in (a_1, a_2)$ is located at the middle of the supporting interval, i.e. $a_M = \frac{a_1+a_2}{2}$. Then substituting this value into (1.12) gives

$$\mathbf{A} \triangleq \mu_{\mathbf{A}}(x) = \begin{cases} 2\frac{x-a_1}{a_2-a_1} & \text{for } a_1 \le x \le \frac{a_1+a_2}{2}, \\ 2\frac{x-a_2}{a_1-a_2} & \text{for } \frac{a_1+a_2}{2} \le x \le a_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

We say that (1.13) represents a *central triangular fuzzy number* (see Fig. 1.18(a)). Similarly to the piecewise-quadratic fuzzy number, it is very suitable to describe the word *close* (*close* to a_M).

Triangular numbers are very often used in the applications (fuzzy controllers, managerial decision making, business and finance, social sciences, etc.). They have a membership function consisting of two linear segments A^l (left) and A^r (right) joined at the peak $(a_M, 1)$ (see Fig. 1.17) which makes graphical representations and operations with triangular numbers very simple. Also it is important that they can be constructed easily on the basis of little information.

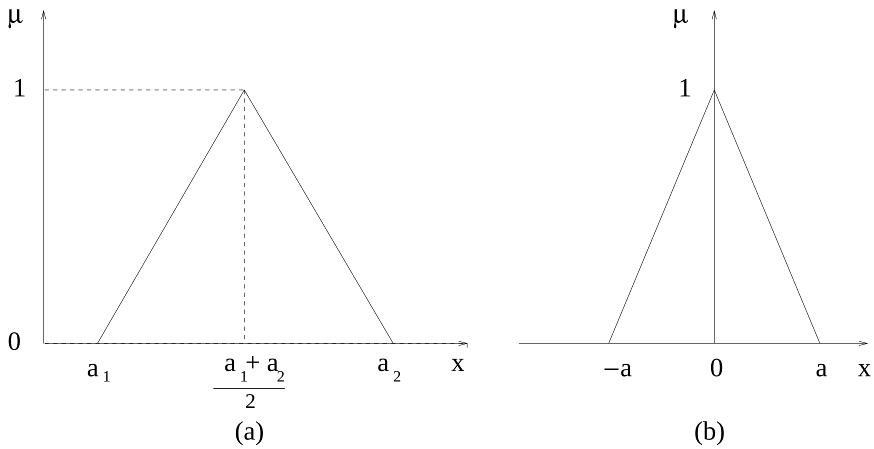


Fig. 1.18. (a) Central triangular number; (b) Central triangular number symmetrical about μ.

Assume while dealing with an uncertain value we are able to specify the smallest and largest possible values, i.e. the supporting interval $A = [a_1, a_2]$. If further we can indicate a value a_M in $[a_1, a_2]$ as most plausible to represent the uncertain value, then the peak will be the point $(a_M, 1)$. Hence with the three values a_1, a_2 and a_M, one can construct a triangular number and write down its membership function (1.12). That is why the triangular number is also denoted by

$$\mathbf{A} = (a_1, a_M, a_2). \tag{1.14}$$

A central triangular number is *symmetrical* with respect to the axis μ if in (1.13) $a_1 = -a, a_2 = a$, hence $a_M = 0$ (see Fig. 1.18(b)). According to (1.14) it is denoted by

$$\mathbf{A} = (-a, 0, a).$$

It is very suitable to express the word *small*. The right branch (segment) of $\mathbf{A} = (-a, 0, a)$, i.e. when $0 \leq x \leq a$, can be used to describe *positive small* (**PS**), for instance *young age, small profit, small risk*, etc. We can denote it by $\mathbf{A}^r = (0, 0, a)$.

More generally, the left and right branches of the triangular number (1.14) can be denoted correspondingly by $\mathbf{A}^l = (a_1, a_M, a_M)$ and $\mathbf{A}^r = (a_M, a_M, a_2)$. They will be considered as triangular numbers and called correspondingly *left* and *right triangular numbers*. The left triangular number $\mathbf{A}^l$ (see Fig. 1.17) is suitable to represent *positive large* (**PL**) or words with similar meaning, for instance *old age*, *big profit*, *high risk*, etc. provided that a_M is large number.

1.6 Trapezoidal Fuzzy Numbers

A *trapezoidal fuzzy number* $\mathbf{A}$ or shortly trapezoidal number (see Fig. 1.19) is defined on R by

$$\mathbf{A} \triangleq \mu_{\mathbf{A}}(x) = \begin{cases} \frac{x-a_1}{b_1-a_1} & \text{for } a_1 \leq x \leq b_1, \\ 1 & \text{for } b_1 \leq x \leq b_2, \\ \frac{x-a_2}{b_2-a_2} & \text{for } b_2 \leq x \leq a_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

It is a particular case of a fuzzy number with a flat.

The supporting interval is $\mathbf{A} = [a_1, a_2]$ and the flat segment on level $\alpha = 1$ has projection $[b_1, b_2]$ on the x-axis. With the four values a_1, a_2, b_1, and b_2, we can construct the trapezoidal number (1.15). It can be denoted by

$$\mathbf{A} = (a_1, b_1, b_2, a_2). \tag{1.16}$$

If $b_1 = b_2 = a_M$, the trapezoidal number reduces to a triangular fuzzy number and is denoted by (a_1, a_M, a_M, a_2). Hence a triangular number (a_1, a_M, a_2) can be written in the form of a trapezoidal number, i.e. $(a_1, a_M, a_2) = (a_1, a_M, a_M, a_2)$.

If $[a_1, b_1] = [b_2, a_2]$, the trapezoidal number is *symmetrical* with respect to the line $x = \frac{1}{2}(b_1 + b_2)$ (see Fig. 1.20). It is in *central* form and represents the interval $[b_1, b_2]$ and real number close to this interval.

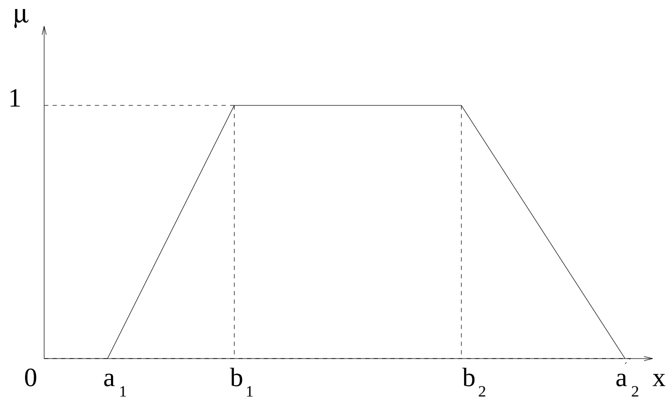


Fig. 1.19. Trapezoidal fuzzy number.

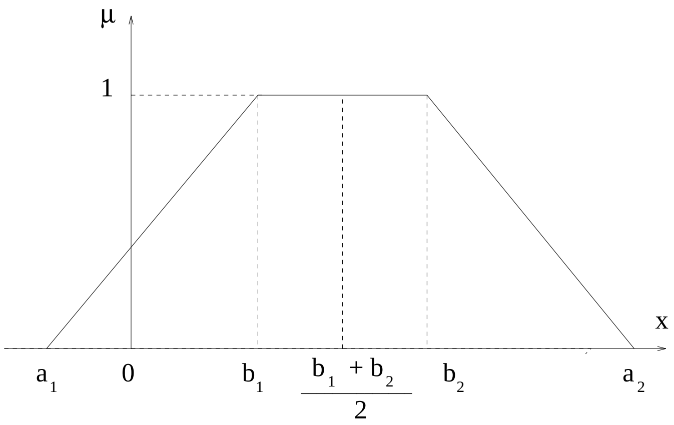


Fig. 1.20. Trapezoidal number in central form.

Similarly to right and left triangular numbers (Section 1.5) we can introduce right and left trapezoidal numbers as parts of a trapezoidal number.

The right trapezoidal number denoted $\mathbf{A}^r = (b_1, b_1, b_2, a_2)$ has supporting interval $[b_1, a_2]$ and the left denoted $\mathbf{A}^l = (a_1, b_1, b_2, b_2)$ has supporting interval $[a_1, b_2]$. Especially they are suitable to represent *small* $\triangleq \mathbf{A}^r = (0, 0, b_2, a_2)$ (Fig. 1.21(a)) and *large* $\triangleq \mathbf{A}^l = (a_1, b_1, b_2, b_2)$ where b_1 is a large number (Fig. 1.21(b)).

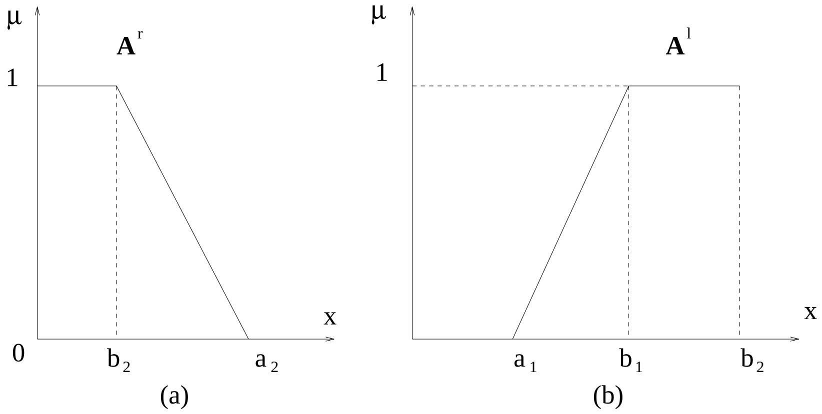


Fig. 1.21 (a) Right trapezoidal number $\mathbf{A}^r$ representing *small*; (b) Left trapezoidal number $\mathbf{A}^l$ representing *large*.

1.7 Fuzzy Relations

Definition of Fuzzy Relation

Consider the Cartesian product

$$A \times B = \{(x, y) \quad | \quad x \in A, y \in B\},$$

where A and B are subsets of the universal sets U_1 and U_2, respectively.

A fuzzy relation on $A \times B$ denoted by $\mathcal{R}$ or $\mathcal{R}(x, y)$ is define as the set

$$\mathcal{R} = \{((x, y), \mu_{\mathcal{R}}(x, y)) | (x, y) \in A \times B, \mu_{\mathcal{R}}(x, y) \in [0, 1]\}, \qquad (1.17)$$

where $\mu_{\mathcal{R}}(x, y)$ is a function in two variables called *membership function*. It gives the degree of membership of the ordered pair (x, y) in $\mathcal{R}$ associating with each pair (x, y) in $A \times B$ a real number in the interval [0, 1]. The degree of membership indicates the degree to which x is in relation with y. We assume that $\mu_R(x, y)$ is piecewise continuous or discrete in the domain $A \times B$; it describes a surface. Formally, the fuzzy relation $\mathcal{R}$ is a classical trinary relation; it is a set of ordered triples.

The definition (1.17) is a generalization of definition (1.6) for fuzzy set from two-dimensional space $(x, \mu_A(x))$ to three-dimensional space $(x, y, \mu_{\mathcal{A}}(x, y))$.[6] Here we also identify a relation with its membership function.

The fuzzy relation in comparison to the classical relation possesses stronger expressive power while relating x and y due to the membership function $\mu_{\mathcal{R}}(x, y)$ which assigns specific values (grades) to each pair (x, y).

Common *linguistic relations* that can be described by appropriate fuzzy relations are: x is *much greater than* y, x *is close to* y, x is *relevant to* y, x and y are *almost equal*, x and y are *very far*, etc.

Example 1.12

Consider the fuzzy relation which consists of finite number of ordered pairs,

$$\begin{aligned} \mathcal{R} \;\; = \;\; & \{((x_1, y_1), 0), ((x_1, y_2), 0.1), ((x_1, y_3, 0.2), \\ & ((x_2, y_1, 0.7), ((x_2, y_2, 0.2_{,}((x_2, y_3, 0.3), \\ & ((x_3, y_1), 1), (x_3, y_2), 0.6), ((x_3, y_3), 0.2))\}; \end{aligned}$$

it can be described also by the table (or matrix)

$$\mathcal{R} \triangleq \begin{array}{c|ccc} {}_{x}\;\;{}^{y} & y_1 & y_2 & y_3 \\ \hline x_1 & 0 & 0.1 & 0.2 \\ x_2 & 0.7 & 0.2 & 0.3 \\ x_3 & 1 & 0.6 & 0.2 \end{array}$$

where the numbers in the cells located at the intersection of rows and columns are the values of the membership function:

$$\begin{aligned} &\mu_R(x_1, y_1) = 0, \quad \mu_R(x_1, y_2) = 0.1, \quad \mu_R(x_1, y_3) = 0.2, \\ &\mu_R(x_2, y_1) = 0.7, \quad \mu_R(x_2, y_2) = 0.2, \quad \mu_R(x_2, y_3) = 0.3, \\ &\mu_R(x_3, y_1) = 1, \quad \mu_R(x_3, y_2) = 0.6, \quad \mu_R(x_3, y_3) = 0.2. \end{aligned}$$

Assuming that $x_1 = 1, x_2 = 2, x_3 = 3, y_1 = 1, y_2 = 2, y_3 = 3$, we can present schematically $\mathcal{R}$ by points in the three-dimensional space (x, y, μ) (see Fig. 1.22).

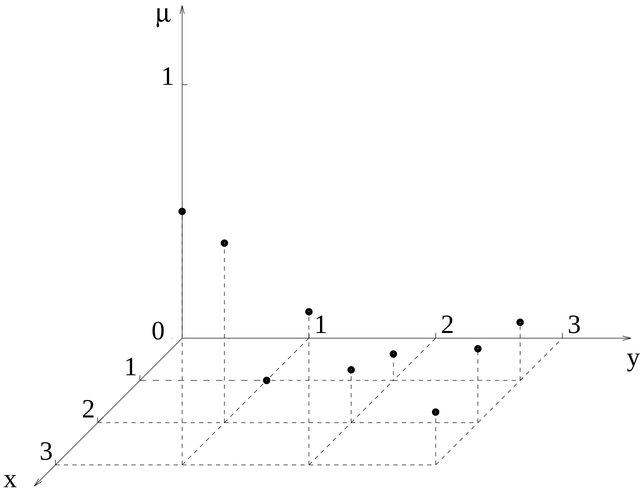


Fig. 1.22. Fuzzy relation $\mathcal{R}$ describing x is greater than y.

Since the values of the membership function 0.7, 1, 0.6 in the direction of x below the major diagonal (0, 0.2, 0.2) in the table are greater than those above in the direction of y, 0.1, 0.2, 0.3, we say that the relation $\mathcal{R}$ describes x *is greater than* y.

The fuzzy relation $\mathcal{R}$ can be expressed also as a fuzzy graph (Fig. 1.23). The numbers at the segments are the degrees of membership.

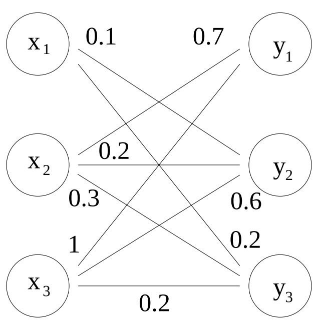


Fig. 1.23. Fuzzy relation $\mathcal{R}$ presented as a fuzzy graph.

□

Example 1.13

Consider the following two sets whose elements are business companies: $A = \{\text{company } a_1, \text{company } a_2, \text{company } a_3\}$, $B = \{\text{company } b_1, \text{company } b_2\}$. Let $\mathcal{R}$ be a fuzzy relation between the two sets that represents the linguistic relation *very far* concerning distance between companies:

$$\begin{aligned}\mathcal{R} \;=\; &\{((\text{company}a_1, \text{company}b_1), 0.9),\\ &((\text{company}a_1, \text{company}b_2), 0.6),\\ &((\text{company}a_2, \text{company}b_1), 1),\\ &((\text{company}a_2, \text{company}b_2), 0.4),\\ &((\text{company}a_3, \text{company}b_1), 0.5),\\ &((\text{company}a_3, \text{company}b_2), 0.1)\}.\end{aligned}$$

The relation can also be presented by the table

$\mathcal{R} \triangleq$

	company b_1	company b_2
company a_1	0.9	0.6
company a_2	1	0.4
company a_3	0.5	0.1

The membership values indicate to what degree the corresponding companies are *very far* from each other. For instance, company a_2 and company b_1 are very far (degree of membership 1) while companies a_3 and b_2 are not very far (degree of membership 0.1).

□

1.8 Basic Operations on Fuzzy Relations

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two fuzzy relations on $A \times B$,

$$\begin{aligned}\mathcal{R}_1 &= \{((x, y), \mu_{\mathcal{R}_1}(x, y))\}, \qquad (x, y) \in A \times B,\\ \mathcal{R}_2 &= \{((x, y), \mu_{\mathcal{R}_2}(x, y))\}, \qquad (x, y) \in A \times B.\end{aligned}$$

We use the membership functions $\mu_{\mathcal{R}_1}(x, y)$ and $\mu_{\mathcal{R}_2}(x, y)$ in order to introduce operations with $\mathcal{R}_1$ and $\mathcal{R}_2$ similarly to operations with fuzzy sets in Section 1.3.

Equality

$\mathcal{R}_1 = \mathcal{R}_2$ if and only if for every pair $(x, y) \in A \times B$,

$$\mu_{\mathcal{R}_1}(x, y) = \mu_{\mathcal{R}_2}(x, y).$$

Inclusion

If for every pair $(x, y) \in A \times B$,

$$\mu_{\mathcal{R}_1}(x, y) \leq \mu_{\mathcal{R}_2}(x, y),$$

the relation $\mathcal{R}_1$ is *included* in $\mathcal{R}_2$ or $\mathcal{R}_2$ is larger than $\mathcal{R}_1$, denoted by $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

If $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and in addition if for at least one pair (x, y),

$$\mu_{\mathcal{R}_1}(x, y) < \mu_{\mathcal{R}_2}(x, y),$$

then we have the proper inclusion $\mathcal{R}_1 \subset \mathcal{R}_2$.

Complementation

The *complement* of a relation $\mathcal{R}$, denoted by $\overline{\mathcal{R}}$, is defined by

$$\mu_{\overline{\mathcal{R}}}(x, y) = 1 - \mu_{\mathcal{R}}(x, y), \tag{1.18}$$

which must be valid for any pair $(x, y) \in A \times B$.

Intersection

The *intersection* of $\mathcal{R}_1$ and $\mathcal{R}_2$ denoted $\mathcal{R}_1 \bigcap \mathcal{R}_2$ is defined by

$$\mu_{\mathcal{R}_1 \cap \mathcal{R}_2}(x, y) = \min\{\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(x, y)\}, \quad (x, y) \in A \times B. \tag{1.19}$$

Union

The *union* of $\mathcal{R}_1$ and $\mathcal{R}_2$ denoted $\mathcal{R}_1 \bigcup \mathcal{R}_2$ is defined by

$$\mu_{\mathcal{R}_1 \cup \mathcal{R}_2}(x, y) = \max\{\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(x, y)\}, \quad (x, y) \in A \times B. \tag{1.20}$$

The operations intersection and union are illustrated in the following example.

Example 1.14

Consider the relations $\mathcal{R}_1$ and $\mathcal{R}_2$ given by the tables

$$\mathcal{R}_1 \triangleq \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0 & 0.1 & 0.2 \\ x_2 & 0 & 0.7 & 0.3 \\ x_3 & 0.2 & 0.8 & 1 \end{array} \qquad \mathcal{R}_2 \triangleq \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.3 & 0.2 \\ x_2 & 0.5 & 0 & 1 \\ x_3 & 0.7 & 0.3 & 0.1 \end{array}$$

Using definitions (1.19) and (1.20) for each ordered pair $(x_i, y_j), i, j = 1, 2, 3,$ gives

$$\mathcal{R}_1 \cap \mathcal{R}_2 \triangleq \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0 & 0.1 & 0.2 \\ x_2 & 0 & 0 & 0.3 \\ x_3 & 0.2 & 0.3 & 0.1 \end{array}; \qquad \mathcal{R}_1 \cup \mathcal{R}_2 \triangleq \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.3 & 0.2 \\ x_2 & 0.5 & 0.7 & 1 \\ x_3 & 0.7 & 0.8 & 1 \end{array}$$

A comparison between the corresponding membership values in $\mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cup \mathcal{R}_2$ shows that $\mathcal{R}_1 \cap \mathcal{R}_2 \subset \mathcal{R}_1 \cup \mathcal{R}_2$ (proper inclusion).

□

Direct Product

Consider the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)), \quad \mu_{\mathcal{A}}(x) \in [0, 1]\},$$

$$\mathcal{B} = \{(y, \mu_{\mathcal{B}}(y)), \quad \mu_{\mathcal{B}}(y) \in [0, 1]\}.$$

defined on $x \in A \subset U_1$ and $y \in B \subset U_2$, correspondingly.

We introduce two types of *direct products* which will be used in the next chapter.

Direct min product of the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ denoted $\mathcal{A} \overset{\times}{\cdot} \mathcal{B}$ with membership functions $\mu_{\mathcal{A} \overset{\times}{\cdot} \mathcal{B}}$ is a fuzzy relation defined by

$$\mathcal{A} \overset{\times}{\cdot} \mathcal{B} = \{(x, y), \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(y)), (x, y) \in A \times B\}, \tag{1.21}$$

which means that we have to perform the Cartesian product $A \times B$ and at each pair (x, y) to attach as membership value the smaller between $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(y)$.

Direct max product of the fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ denoted $\mathcal{A}\dot{\times}\mathcal{B}$ with membership function $\mu_{(\mathcal{A}\dot{\times}\mathcal{B})}(x, y)$ is a fuzzy relation defined by

$$\mathcal{A}\dot{\times}\mathcal{B} = \{(x, y), \max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(y)), (x, y) \in A \times B\}. \tag{1.22}$$

Here each pair (x, y) has for membership value the larger between $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(y)$.

Example 1.15

Given the fuzzy sets

$$\mathcal{A} = \{(x_1, 0), (x_2, 0.1), (x_3, 1)\},$$

$$\mathcal{B} = \{(y_1, 0.3), (y_2, 1), (y_3, 0.2), (y_4, 0.1)\},$$

the direct min product and the direct max product according to (1.21) and (1.22) are the fuzzy relations

$$\mathcal{A} \overset{\times}{\cdot} \mathcal{B} \triangleq \begin{array}{c|cccc} {}_{x}\ {}^{y} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0.1 & 0.1 & 0.1 & 0.1 \\ x_3 & 0.3 & 1 & 0.2 & 0.1 \end{array}$$

$$\mathcal{A}\dot{\times}\mathcal{B} \triangleq \begin{array}{c|cccc} {}_{x}\ {}^{y} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.3 & 1 & 0.2 & 0.1 \\ x_2 & 0.3 & 1 & 0.2 & 0.1 \\ x_3 & 1 & 1 & 1 & 1 \end{array}.$$

□

1.9 Notes

1. The formal development of set theory began in the late 19th century with the work of George Cantor (1845–1918), one of the most original mathematicians in history. Set theory has been used to establish the foundations of mathematics and modern methods of

mathematical proof. Cantor's sets are crisp. Each element under consideration either belongs to a set or it does not; hence there is a line drawn between the elements of the set and those which are not. The boundary of a set is rigid and well defined (see Example 1.5). However in reality things are rather fuzzy than crisp.

2. A paradox coming from ancient Greece has caused serious problems to logicians and mathematicians. Consider a heap of grains of sand. Take a grain and the heap is still there. Take another grain, and another grain, and continue the process. Eventually ten grains are left, then nine, and so on. When one grain is left, what happens with the heap. Is it still a heap? When the last grain is removed and there is nothing, does the heap cease to be a heap? There are many paradoxes of similar nature called "sorites." This word comes from "soros" which is the Greek word for heap. For instance let us apply the above procedure to the cash (say, one million) of a rich person. He/she spends one dollar and is still rich; then another dollar and so on. When one hundred dollars are left, what happens to his/her richness? When does that person cease to be rich? In the crisp set theory such dilemmas are solved by sort of appropriate assumptions (as in Example 1.5) or by decree. In the case of the heap a certain natural number n is to be selected; if the number of sand grains is $\geq n$, then the grains constitute a heap; $n-1$ sand grains does not form a heap anymore. This defies common sense. Also how to select the number n? Is it 100, 1000, or 1,000,000, or larger? Common sense hints that the concept *heap* is a *vague* one. Hence a tool that can deal with vagueness is necessary. The concept of fuzzy set, a generalization of Cantor's sets, is such a tool (see Example 1.7).

   The following thoughts by Bertrand Russell (1923) are quoted very often: "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial one. The law of excluded middle is true when precise symbols are employed but it is not true when symbols are vague, as, in fact, all symbols are." "All language is vague." "Vagueness, clearly, is a matter of degree."

An important step towards dealing with vagueness was made by the philosopher Max Black (1937) who introduced the concept of vague set.

3. The concept of fuzziness was introduced first in the form of fuzzy sets by Zadeh (1965).

   According to dictionaries (see for instance *Merriam-Webster's Collegiate Dictionary* and *The Heritage Illustrated Dictionary of the English Language*) and also use in everyday language the words *fuzzy*, *vague*, *ambiguous*, *uncertain*, *imprecise*, and their adverbs, are more or less closely related in terms of meaning. This statement is supported by the following brief explanations.

   Fuzzy: not sharply focused, clearly reasoned or expressed; confused; lacking of clarity; blurred.

   Vague: not clearly expressed, defined, or understood; not sharply outlined (hazy); lack of definite form.

   Ambiguous: capable of being understood in two or more possible ways; doubtful or uncertain (synonym: vague).

   Uncertain: not certain to occur; not clearly identified or defined; lack of sureness about something; lack of knowledge about an outcome or result.

   Imprecise: not precise, inexact, vague.

   There are various opinions on the meaning of these words and their use and misuse in common language, philosophy, and in fuzzy logic. We leave it to philosophers and linguistists to debate and deliberate on the subject if they choose to do it. Poper (1979) for instance sounds quite discouraging: "One should never quarrel about words, and never get involved in questions of terminology. One should always keep away from discussing concepts. What we are really interested in, our real problems, are factual problems, or in other words, problems of theories and their truth." There is some truth in Poper although he goes to an extreme. We think it will be useful for the better understanding of this book to provide a clarification.

Fuzzy, adv. *fuzziness*, in fuzzy logic is associated with the concept of graded membership which can be interpreted as degree of truth (see Section 2.6). The objects under study in fuzzy logic admit of degrees expressed by the membership functions of fuzzy sets (see Section 1.2). Problems and events in reality involving components labeled as vague, ambiguous, uncertain, imprecise are considered in this book as fuzzy problems and events if graded membership is the tool for their description. In other words, when gradation is involved, vagueness, ambiguity, uncertainty, imprecision are included into the concept of fuzziness.

Beside the fundamental volume *Fuzzy Sets and Applications: Selected Papers by L.A. Zadeh* (1987), here we list several important books dealing with fuzzy sets and fuzzy logic used in this text: Kaufmann (1975), Dubois and Prade (1980), Zimmermann (1984), Kandel (1986), Klir and Folger (1988), Novák (1989), Terano, Asai, Sugeno (1992).

Fascinating popular books on fuzzy logic are written by McNeill and Freiberger (1993) and Kosko (1993).

4. The notion of fuzzy set is sometimes incorrectly considered as a type of probability. Although there are similarities and links between fuzzy sets and probability, there are also substantial differences. For instance, *grade* or *degree of membership* is not a probablistic concept. In Example 1.8 (*tall men*), a man who is 180 cm tall has a degree of membership 0.78 (or 78%) in the set *tall men*. We can say this person is 78% tall (almost tall), but we can not say that there is a probability of 78% that he is tall.

5. The concept of fuzzy number was introduced after that of fuzzy set. Valuable contributions to fuzzy numbers were made by Nahmias (1977), Dubois and Prade (1978), and Kaufmann and Gupta (1985) (see also G. Bojadziev and M. Bojadziev (1995)).

   In many applications both fuzzy numbers and fuzzy sets can be used equally well although presentations with fuzzy numbers are somewhat simpler. For general studies and also for facilitating *fuzzy logic*, fuzzy set theory is a very suitable tool.

6. Fuzzy relations were introduced by Zadeh (1971) as a generalization of both classical relations and fuzzy sets.